- Mathematics includes arithmetic, but it is more than just arithmetic. Many middle school students have spent years practicing arithmetic algorithms and evaluating algebraic expressions but very little time engaging in the study of geometry, or probability, or data analysis, or functions. Middle school students also need to be constantly reminded that mathematical thinking is mathematics, too.
- There is no direct route to understanding. Mistakes and confusion are an essential part of this process. One of the principles that consistently guides me is this: "We must recognize that partially grasped ideas and periods of confusion are a natural part of the process of developing understanding" (California State Department of Education 1987, 14).
- There are many ways to be good at math. Being fast and accurate with procedures is one way, but other ways include being good wonderers, good visualizers, good explainers, flexible problem solvers, adept pattern finders, or facile technology users. Everyone can be good at mathematics and we need people who are good at all of those different ways in our mathematical community.
- There are many ways to approach most mathematical problems, even those with only one answer. And while some ways may appear to be more efficient than others, what is most important is that everyone has at least one way to solve a problem that she or he really understands. Mathematics should make sense.
- We can get better at the skills of mathematics through practice, but talking and listening to each other (not just the teacher) about mathematical ideas helps us understand mathematical ideas in different ways. We really understand what we can explain.

In these video cases you are offered the opportunity to watch moments of lessons in which the teaching decisions and discourse that ensued were unique to this particular moment in time. We hope you will think about these lessons in the context of your own classroom, your students, and your learning. We hope they will prompt useful questions, ideas, and discussions among professionals on a common journey.



# Chapter 2

# Building on Student Ideas
## The Border Problem, Part 1 (March 9)

*From whence does algebra grow?*
*It grows from the study of growth itself.*

DOSSEY (1997, 20)

*The fact that many children cannot understand much of the algebra we teach leads them to dislike and ultimately reject it, and thence possibly the whole subject of mathematics.*

ORTON AND FROBISHER (1996, 14)

### Background of the Lesson—Cathy's Perspective

It has long been an interest of mine to find ways to make algebra more accessible. So, when I needed to select a writing project for my master's degree, it was natural that I chose algebraic representation as a topic.

The research for my project corroborated my teaching experiences. I learned that far too many students are not successful in algebra, despite its role as a gatekeeper for further mathematics study and college entrance (Silver 1997). I learned that the ideas of equality and variable are particularly thorny obstacles and that our students are often not able to apply the routine skills they have learned to problem-solving situations (Brown et al. 1988). None of these findings came as a surprise. But I also discovered that, while there is no consensus on a single best way to remedy these problems, there is general agreement that an emphasis on pattern generalization supports the development of algebraic thinking (Driscoll 1999; Haimes 1996; Lee 1996; Mason 1996; NCTM 2000; Schoenfeld and Arcavi 1988) and that the concept of function is fundamental to the ideas of algebra (Brenner et al. 1997; Leitzel 1989; Lodholz 1990; Thorpe 1989). Since pattern generalization and a functions approach that focuses on the relationship between quantities (Chazan 2000) are closely related, I decided this was the way I wanted to introduce the ideas of algebra to my seventh graders.

My experiences with using a functions-based approach in my work in professional development raised related issues for me. As my colleagues and I designed lessons based on the visualization of growth patterns for teachers in our summer workshops, I began adapting those lessons for my middle school students. To my dismay, my students, like many of the teachers in our workshops, did not have the powerful experiences I had expected. While they enjoyed the tasks and successfully completed tables of values, they faltered when they tried to write algebraic expressions to represent the relationships. The language of algebra seemed foreign and artificial; the teachers, in particular, seemed to be trying to remember what to do rather than allowing the algebraic representation to emerge from what they understood about the relationships they were investigating. This experience prompted my personal quest to learn more about how to help bring understanding to the complex world of generalization. I kept tinkering with the tasks and trying different approaches. I went to conferences, read articles and books, and talked with colleagues who were struggling with the same issues.

I came to believe that finding the algebraic rule really isn't the point. While there are many techniques to help students find an algebraic rule, Thornton (2001) argues that it is less important that students be able to find the algebraic rule than that they recognize that the different visualizations of a pattern can be described symbolically in equivalent algebraic expressions. The underlying complexity of the mathematics embedded in the problem situation can become lost—for both the teacher and the student—when finding the rule becomes the point of the exercise. As Duckworth (1991) notes in relation to poetry, people tend to notice very different things, and each of those things contributes to greater understanding for each person. "A teacher who presents a subject matter in all of its complexity makes it more accessible by opening a multiplicity of paths into it" (8–9).

Fortified by these convictions, I designed a sequence of linked lessons in which students would investigate growth patterns in order to build their understanding of algebraic representation. The lessons consisted of interesting and predictable growth patterns—all discrete polynomial functions—which the students would investigate using cubes, tiles, toothpicks, pattern blocks, and Cuisenaire Rods®. These lessons addressed the following National Council of Teachers of mathematics (NCTM) content standards, which are particularly relevant for the middle grades:

- Develop an initial conceptual understanding of different uses of variables
- Develop an initial conceptual understanding of the notion of function



- Recognize and generate equivalent forms for algebraic expressions
- Represent, analyze, and generalize a variety of linear and nonlinear functions with tables, graphs, verbal rules, and symbolic rules
- Relate and compare different representations of a function

If lists of content standards were sufficient, however, children's difficulties with algebra would have been solved long ago. I thought long and hard not only about what problems the students would work on, but about how to make my instruction yield the most benefit for students' algebraic thinking. How any activity is enacted in a classroom—and what the students learn from it—depends not only on the task itself but on the teacher's image of the essential mathematics in the task (Thomson et al. 1994). So, as I planned to teach these lessons, I thought carefully about how I would pose the problems and what teaching strategies would be the most effective in maintaining a healthy balance between allowing students to pursue their own ways of thinking and providing information to support the development of symbolic mathematics (Hiebert et al. 1997).

The Border Problem is the first lesson in my unit. Versions of this classic problem appear in many curriculum materials in which algebra is being introduced (Burns and Humphreys 1990; Fendel et al. 1997; Lappan et al. 1998b). In this problem, students are presented with some version of the diagram in Figure 2–1, in this case a 10-by-10 grid, with its "border" colored.

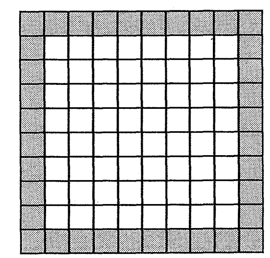


**Figure 2–1.**

Students are asked to calculate the number of colored squares in the border. Since there are six different ways (other than counting) to visualize how to find the number of squares in the border, the Border Problem has great potential for examining equivalent expressions. Also, since the relationship between the number of unit squares on one side of the grid and the number of unit squares in the border is a function, this problem also contains promise for establishing other important notions from which major algebraic ideas can be developed. These include the following:

- The arithmetic that emerges from the geometry of the border can be applied when the grid changes size.

- These generalized calculations can be described verbally, pictured geometrically, expressed symbolically, and represented graphically.
- The symbolic expressions that represent the particular visualizations from which they have emerged are algebraically equivalent.

I had had countless discussions with colleagues about the best way to draw out the potential in this problem. I knew what I was *not* trying to develop: I did not want to focus on simplifying expressions (yet), nor did I want students to determine the "most efficient" way of finding the number of squares in the border (efficiency is often in the eye of the beholder when it comes to mathematical thinking). That there were thirty-six squares in the border was certainly not the point, as the squares could be counted easily by any second grader. I knew that I did not want to do the students' thinking for them by showing different methods to calculate the number of squares in the border. Deborah Ball says, "With my ears to the ground, listening to my students, my eyes are focused on the mathematical horizon" (1993, 376). I set out to draw upon students' natural abilities to visualize and, in doing so, to breathe life into symbolic representation.

The Border Problem typically takes three forty-five-minute class periods. On the first day, the students generate a variety of ways to figure out the number of unit squares in the border. Their geometric visualizations serve as the "currency" (Hiebert et al. 1997) with which they will eventually build connections between the arithmetic and geometry of the function. They also are asked to imagine the grid changing size (either stretching or shrinking) with an eye toward generalization.

The purpose of the second class session is to introduce the tools of multiple representations. Using one visual method as a model, the students learn to represent that method geometrically, arithmetically, verbally, and algebraically. This is examined in more detail in Chapter 3. Finally, on the third day of their work on this problem, students choose another method and work in pairs to represent that method using the four representations they have learned.

This case begins as I introduce the problem; the segment continues as students generate and geometrically justify different calculation methods.

## Watch "Building on Student Ideas: The Border Problem, Part 1," CD 1

### Lesson Analysis and Reflection

As I put the slide of the grid on the overhead projector, I told the students that it was 10 by 10 and asked them to "figure out *without talking*, *without writing*, and *without counting one by one*" how many unit squares were in the colored portion.



Why without talking? I wanted to have as wide a variety of methods as possible, and sharing methods early on might limit the variety.

Why without writing? The numbers are well within the grasp of seventh graders, and freedom from pencils helps students enlist the power of visualization as a support to calculation.

Why without counting one by one? Although this can yield a correct answer, it is not a method that lends itself to the goals of pattern generalization. (As it turned out, one student did go to the overhead and count around one by one but ended up with the wrong answer!)

Why didn't I give them each a grid to facilitate their thinking? I have found that if students have a grid, they tend to count one by one. If counting one by one is their initial method, then sharing other ways to find the number of squares is an intellectual or creative enterprise without meaning.

My next decision was to have the students talk to each other just about what they got. In saying this, I meant that I first wanted them to talk about the answers they got, not yet about how they got them. I knew from experience that thirty-eight and forty are predictable answers because of compensation errors, and while I did not want the answer to divert us from the main point of the lesson, I did want to shine a light on the logic of these common errors. I use this teaching strategy to help students see mathematics as a sense-making activity and to show that we can use errors as sites for learning (Hiebert et al. 1997).

After this, students volunteered to share how they got thirty-six. I recorded their methods.

Why did I record, rather than let students do so? I wanted to model correct numerical representation for instructional advantage. I recorded all of the arithmetic expressions *horizontally* so that they could more easily be connected to the algebraic expression. I also avoided use of the equals sign; I did this to counteract the widespread interpretation of the equals sign as a "get the answer" operator and to emphasize the idea that the numerical expression represents an answer in the same way that a symbolic expression does. A student who thinks the equals sign means "write the answer" is likely to think that, for an expression like $x + y$, a single-term answer is required (e.g., $x + y = z$) (Booth 1988). I also intended to name a method after its originator; this would help distinguish one method from another in the lesson that would follow—and besides, it is nice to give credit to students for their ways of thinking.

Sharmeen was first to share; she saw the border as $4 \times 10 - 4$ (four sides of ten unit squares each minus four overlapping squares at the corners). I wish I had asked her to go to the overhead projector and show how her arithmetic made sense geometrically. Not doing so was one of those odd oversights that occur, no matter how much I think about and plan a lesson. Pressing Sharmeen to distinguish between the two 4s in her arithmetic expression, however, was a way of helping everyone in the class think about how the two 4s played a different role in the numerical expression, and it helped emphasize the connection between the numerical expression and its geometric foundation.

Colin's method was to add 10 + 9 + 9 + 8, visualizing first one full side, then a side without one corner, and another, and the last side without two corners. In his explanation to the class, Colin not only described what he had done but justified geometrically each of the numbers in his expression. The intentional connection between arithmetic and geometry moved toward building a common understanding that whenever we make a statement about a pattern, we justify it geometrically. This idea would be foundational for all of the problems in this unit.

Joe volunteered the method that had been employed by most of the students in the class: 10 + 10 + 8 + 8. He had counted the full top and bottom of the border, and then had subtracted two from each of the sides to account for the overlaps at the four corners. When Joe was explaining his method, I heard him say that he had added "those two" (meaning 8 + 8) and then added that result to twenty. If I had recorded it *exactly* as he had said it, I would have recorded:

$$10 + 10 = 20$$
$$8 + 8 = 16$$
$$20 + 16 = 36$$

I did not do so because of past experience. Too many students, using the logic that a different number should represent a different letter, had attempted to generalize this same method like this:

$$a + a = x$$
$$b + b = y$$
$$x + y = z$$

I therefore deliberately wrote Joe's method as a single expression (10 + 10 + 8 + 8) rather than three equations. I made sure to check with him, however, to confirm that this was a fair representation of his method.

Melissa used a subtraction method to find the number of squares in the border. She saw two squares: one, a 10-by-10 grid, and the other, an 8-by-8 grid. The border was the difference of the two. I recorded Melissa's method carefully, writing $(10 \times 10) - (8 \times 8)$. Even though the parentheses were not required, I included them to emphasize the subtraction of two numbers, one-hundred and sixty-four.

Tina, who had seen the border as four overlapping lengths of nine, reported $4 \times 9$ as her method.





At this point in the lesson, one student suggested "six times six" as a method. This is a common dilemma with students who are new to classes that build on student thinking. I had taken over this class for another teacher in January and, this being early March, some students had not yet internalized the notion that mathematical sense making was essential. This student assumed that our aim here was to gather as many methods as possible, regardless of whether the methods made sense. I had a brief discussion about how, while $6 \times 6$ did indeed equal the number of squares in the border, we would not record it as a method since it did not emerge from the geometry of the border.

After the students had generated all of the methods they could think of, there was still one additional method $(4 \times 8 + 4)$ that they would need for the next day's lesson. In the past I had dealt with this by asking students to think about "how else someone might see it," but this time I decided to tell them the method (saying that a student in another class, named Zach, had come up with it) and see if they could figure out why it made sense geometrically. This is an example of what Mark Driscoll, in his book *Fostering Algebraic Thinking,* calls "doing-undoing," a strategy he believes is "critical to developing power in algebraic thinking" (1999, 1). This approach proved to be interesting and engaging for this class and I was delighted that Kayla, who often lacked confidence, was eager to explain Zach's method.

With all of the methods now on the board, I had planned to have students think about changing the size of the grid. But Kay's striking observation that Tina's and Zach's methods were alike changed the course of my lesson, taking it to a deeper level of analysis. My original plan had been to have the students mentally stretch and shrink the grid, but making connections among methods and how they are alike and different begins to lay the foundation for the concept of algebraic equivalence. In other words, the numerical expressions are equivalent not only because they generate the same number of squares but because their *structures* have similarities and differences that compensate for each other. Kay's comment was one of those gifts to classroom interaction that turn up more frequently as students gain faith in their own reasoning, and it triggered similar observations about the other methods.

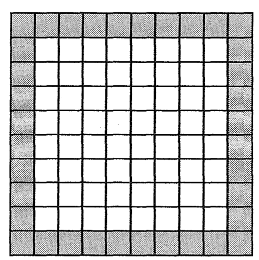
After thinking about connections among all of the methods on the board, there was still time for the students to think about changing the size of the square. I was momentarily uncertain whether the students should all use a particular method or each use their own method; while each person's own method might be more accessible, a single approach would be easier to discuss as a class. I decided on accessibility, asking students to use their own method to visualize a 6-by-6 grid. I anticipated that there would be clusters of students who had used each method. Sharmeen's method, $4 \times 10 - 4$, was the first on our list of methods, so I asked the class, "What would Sharmeen have done?" as a way to help students apply her method to any size grid—a stepping-stone to our work of generalization.

After the video selection, Shelley, the first one to answer this question, initially made a mistake in the numbers she used, saying "Six times ten minus six" and then "four times six minus six." It was evident that Shelley was looking at number patterns rather than the geometry of the method:

| | | |
|---|---|---|
| 10 by 10: | Sharmeen: | $4 \times 10 - 4$ |
| 6 by 6: | Shelley, first try: | $6 \times 10 - 6$ |
| | Shelley, second try: | $4 \times 6 - 6$ |
| | Shelley, final explanation: | $4 \times 6 - 4$ |

I did not correct Shelley because I was more interested in letting her figure out her own error. Many of the students were eager to correct her, but when she started to explain the reasoning behind $6 \times 10 - 6$, she changed her idea to $4 \times 6 - 6$. This prompted her classmates to raise their hands again. But when Shelley began to justify $4 \times 6 - 6$, she again changed her answer, this time to $4 \times 6 - 4$ (for the corners). This demonstrates how student errors illuminate student thinking and how important it is for people to have the opportunity to correct their own mistakes. The period was nearly over, but I planned to revisit this problem in more depth the next day.

## Case Commentary—Jo's Analysis



### *The Implementation of Tasks*

"What I would like you to do mentally is figure out, without counting one by one, how many squares are in the colored portion. How many unit squares—and without talking, without counting one by one, without writing it."

These are the words with which Cathy started the lesson. An uninformed observer might think that such precise instructions are not needed—why didn't Cathy just ask the students to work out how many squares were in the border?





Why did she show the border on the overhead and not give the students their own copies to work from? But Cathy's words and actions were carefully planned and they were important for encouraging the many methods and connections that emerged. Another teacher once watched the case and saw the wonderful work the students were doing and the excitement they had in seeing different methods. She decided to try the activity in her classroom, but she was teaching a group of students that she perceived as weak, so she gave them their own copies of the border to work with. When the teacher asked the students how many squares were on the border, they looked at their papers and counted. The teacher went on to ask the students for their methods for working out the number of squares, but nobody had any methods to share and the students couldn't see the point of the activity. This event illustrated for me the huge importance of *task implementation*—the decisions teachers make as they present tasks in the classroom. Stein, Smith, Henningsen, and Silver (2000) wrote an important book in which they presented different cases of task implementation. They showed that the same task could be implemented at a number of different levels, and that frequently teachers use open tasks with the hope that they will inspire student thinking, but close the tasks down to the point that students can answer them with minimal cognitive effort. In Stein and her colleagues' words, the cognitive demand of tasks is frequently lowered. Cathy carefully chose the words and actions with which she started the lesson to maintain a high level of thinking and to generate the wonderful range of methods the students produced in the case.

### *Detailed Planning*

How did a task that may appear trivial on paper explode into such a broad array of methods and unexpectedly complex connections? This came about, in part, because Cathy had planned so carefully and she was purposefully using the task as a means to generate methods that could be expressed algebraically and that could lead into discussions of equivalence. The mathematical meanings that were developed in the first two cases (and more generally in the entire unit of work) demonstrate the importance of planning across time—of developing a series of linked lessons that purposefully introduce students to ideas and then build upon them. Some people may look at this case and think it is a nice example of students sharing ideas and methods, but it is more than that. It is also the start of a mathematical journey that will take students into the realms of algebraic equivalence and the representation of functions. The case may also look like an instance of students sharing any methods they happened to use, but it is not; it is a teacher purposefully drawing out six particular different methods. The fact that Cathy was able to look for those methods, and even bring in another method that had not been raised by her class and ask students to work "backwards" (Driscoll 1999) to visualize it, was testament to her careful planning of the lessons (Ball 1993).

### *Connecting Representations*

A number of interesting events transpired in the course of the lesson. One moment I find particularly significant was when Cathy showed Zach's method and asked the students why it made sense with the diagram. As students looked at Zach's method and at the grid, we started to hear choruses of "Ooh," "Aah," and "Oh, I see it!" Those are the sounds of mathematical wonder—the students were curious about the way Zach's method worked, they were pleased when they could see it, and they experienced some of the joy of a mathematical breakthrough. These are the sorts of moments that mathematics classrooms should be full of, but they rarely happen. In the quote that opens this book, Margaret Wertheim's (1997) description of the "secret image" of pi, the "treasure of the universe that had just been revealed," reminds us both of the mystery and intrigue of mathematics and the natural wonder of children. Classrooms should be full of such moments, as we pique children's interest and surprise them with the connections and relationships that make mathematics so special. Since it is unlikely that students will experience such insights by plowing through worksheets that require them to repeat procedures, we must create such moments in more engaging and interactive ways. Seeing a mathematical expression represented in a diagram should not be a rare source of wonder for students, they should receive many opportunities to explore the relationships between geometric and algebraic forms. For some students it is a source of surprise that such connections can even be made. In this classroom the expressions of "Ooh," "Aah," and "Cool" as students considered the different representations of the border told us that they had experienced some of the mathematical insight and wonder that Wertheim so powerfully describes.

### *Connecting Different Methods*

Another interesting moment in the lesson came when Kay commented that Zach's method was like Tina's, "only that (it was) eight times four and not nine times four." This led into a discussion in which students compared and contrasted the different methods. Kay demonstrated in that moment the classroom practice she had learned of looking for the connections among different methods. This is an extremely important mathematical practice. After the students discussed the connections among methods, Cathy asked them to shrink the square in their minds and to use one of the methods to find the number of squares in the border of the new square. This is an interesting task as it can be solved only by linking the particular mathematical expression to the imagined square, which involves high levels of visualization and reasoning. This act of connecting a prealgebraic expression to a geometric representation is extremely valuable. Mark Driscoll (1999) talks about the importance of appreciating the *form* of algebraic expressions, as algebraic expressions of different forms give insights into different aspects of mathematical relationships. For example, the expression $4 \times n - 4$ for representing the squares in the border of a grid with side length $n$ shows something different than the expression $n + (n - 1) + (n - 1) + (n - 2)$. If the value of different expressions is partly determined by the insights they show, then their form becomes critically important. This task is cultivating an appreciation of algebraic form.

Similarly, earlier in the lesson, when students were looking to see where certain numbers were represented on the diagram, they were being encouraged to appreciate that algebra is a *tool* for representing a mathematical relationship, rather than a result. Noss, Healy, and Hoyles (1997) point out that somewhere along the way, we stopped viewing algebra as a set of problem-solving tools and now regard it as an end point. Fiori (2004) makes a similar point, noting an interesting reversal of cause and effect: problems have become a resource for practicing algebra, instead of algebra being learned and used as a tool for exploring problems.

The students in this class had an interesting day—they saw that mathematical expressions represent something real, and they marveled at the links between written expressions and visual representations. Their appreciation of the mathematics was tangible and the scene was set for their first encounter with variables.