BGSU Mathematics Competition
March 21 2015 **A**

No cell phones are allowed. Show all work. Explain your answers.

1) A latice point is a point in the plane with integer coordinates. How many latice points (x, y) satisfy $x^2 + y^2 \leq 15$?

2) Use the picture on the back page to decipher the following message:

EBOX EHGZ TGW IKHLIXK

3) Prove $e^\pi > \pi^e$, where $e = 2.71828\cdots$ is the limit of $(1 + 1/n)^n$ as $n \to \infty$, and $\pi = 3.1415926\cdots$ is the ratio of a circle's circumference of its diameter.

4) Could you always choose out of any 100 numbers 15 numbers in such a way that the difference of any two is divisible by 7?
b) Could you always choose out of any 100 numbers 16 numbers in such a way that the difference of any two is divisible by 7?

5) Prove that for any integer n the number $\frac{n^5 - 5n^3 + 4n}{120}$ is an integer.

6) Let $N = 9 + 99 + 999 + ... + \overbrace{99...9}^{99}$. Determine the sum of digits of N.

7) Suppose that an equilateral triangle is given in the plane, with none of its sides vertical. Let m_1, m_2 and m_3 denote the slopes of the three sides. Show that

$$m_1 m_2 + m_2 m_3 + m_3 m_1 = -3.$$

8) Prove that if a, b and c are odd integeres, then the polynomial $ax^2 + bx + c$ has no rational roots.